Introduction to the equivalence and classification of quadratic sub-manifolds in $T\mathbb{R}^2$

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Overview

Definitions

1. *smooth*: means C^∞ smooth,
2. We consider a *smooth* manifold $\mathcal{X}$ of dimension 2, since all results are local, we can imagine $\mathcal{X}$ an open subset of $\mathbb{R}^2$, equipped with coordinates $x = (z, y)$,
3. $T\mathcal{X}$: the tangent bundle of $\mathcal{X}$, with coordinates $(x, \dot{x})$,
4. Given two vector fields f and g on $\mathcal{X}$ we define their Lie bracket (in coordinates) by $\mathrm{ad}_f g := [f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g \in V^\infty(\mathcal{X})$

We consider a smooth sub-manifold $\mathcal{F} \subset T\mathcal{X}$, locally given by:

$$F(x, \dot{x}) = 0. \qquad (\mathcal{F})$$

We assume that F is smooth and that $\frac{\partial F}{\partial \dot{x}} \neq 0$.

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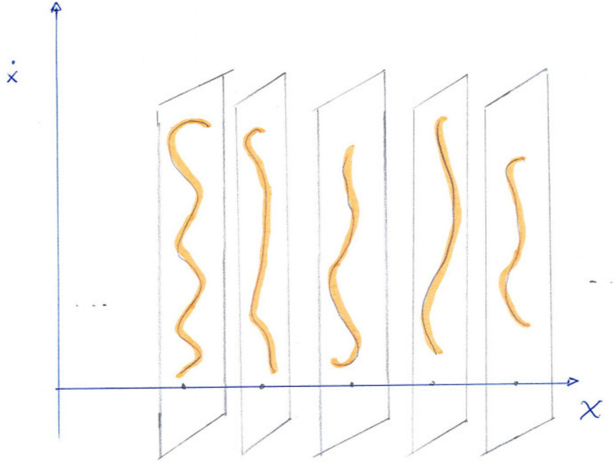
What is it?



$\mathcal{F}: f(x, \dot{x}) = 0$.

Motivations - 1

Quadratic sub-manifolds in Physics are common

From [B, 1991]. Consider the *attitude control problem* for a rigid spacecraft governed by gas jets. Let $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$ be the orientation of the satellite and $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$ be the angular velocity measured in a specific frame attached to the satellite. The control problem is,

$$\begin{cases} \dot{\theta}_1 & = \omega_1 \qquad \dot{\omega}_1 & = a_1\omega_2\omega_3 \\ \dot{\theta}_2 & = \omega_2 \qquad \dot{\omega}_2 & = u_2 \\ \dot{\theta}_3 & = \omega_3 \qquad \dot{\omega}_3 & = u_3 \end{cases}$$

which is the quadratic sub-manifold (in $\mathrm{T}\mathbb{R}^3$) given by $\dot{\omega}_1 = a_1\dot{\theta}_2\dot{\theta}_3$

Motivations - 2

From the mathematical point of view-1

Also in [B, 1991], Bonnard started a classification of quadratic control systems (not of sub-manifolds), he left very interesting questions to answer.

From the mathematical point of view-2

The following sub-manifold appear in [A-N-N, 2015] as infinitesimal realisation of the simple Lie Algebra $\mathfrak{so}(r+2, s+2)$ (where (r, s) is the signature of k).

$$\dot{z} = \frac{1}{2}\sum_{ij}^{m} k_{ij}\dot{y}^i\dot{y}^j.$$

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Equivalence of sub-manifolds

We act on sub-manifolds by smooth diffeomorphisms, $\tilde{x} = \phi(x)$ and we say that two sub-manifold $\mathcal{F}$ and $\tilde{\mathcal{F}}$, given by F and $\tilde{F}$, are (locally)-equivalent if there exists a (local) diffeomorphism $\tilde{x} = \phi(x)$ such that $F(x, \dot{x}) = \tilde{F}(\phi(x), D\phi(x)\dot{x})$.

The Question

When is a given sub-manifolds equivalent to a linear sub-manifold? Does coordinates x exist such that $\mathcal{F}$ can be written,

$$\omega(\dot{x}) = 0, \quad \omega \in \Lambda^1(\mathbb{R}^2)$$

$$a(x)\dot{z} + b(x)\dot{y} = 0$$

This question is immediately generalised by: when is a sub-manifold equivalent to an affine sub-manifold: $\omega(\dot{x}) + h(x) = 0$.

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Quadratic sub-manifolds

The problem we address today is the equivalence with a so-called *quadratic* sub-manifold:

$$F(x,\dot{x}) = \dot{x}^T g(x)\dot{x} + \omega(\dot{x}) + h(x) \qquad (\mathcal{F}_q)$$

with $g(x)$ a smooth 2 by 2 symmetric matrix with $\operatorname{rk}(g(x)) \geq 1$.

Assumptions: We consider the degenerate case $\operatorname{rk}(g(x)) = 1$ in a neighbourhood. Let $A \in V^\infty(\mathcal{X})$ such that $\ker g = \operatorname{sp}\{A\}$. We assume $\omega(A) \neq 0$ (the most general assumption in our degenerate case).

$$F(x,\dot{x}) = -a(x)\dot{y}^2 + \dot{z} - b(x)\dot{y} - c(x) \qquad (\mathcal{F}_q^1)$$

in suitable coordinates.

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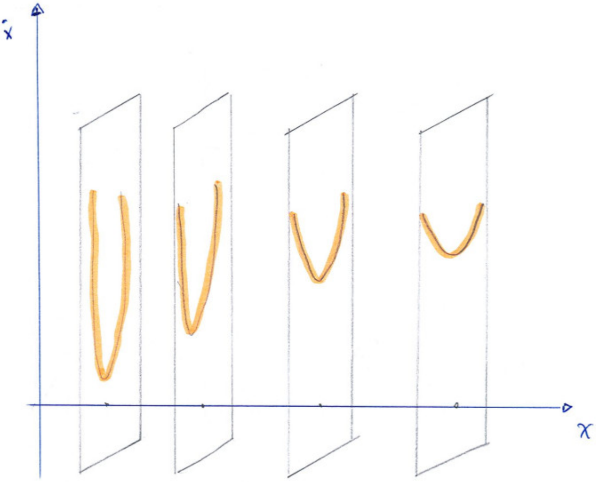

$$f^1_q : \dot z - a(x)\dot y^2 - b(x)\dot y - c(x) = 0 \,.$$

Solving the equivalence problem by feedback equivalence of affine control systems

The idea: prolong twice the sub-manifold and then see it as an affine control system.

$$\mathcal{F}_q^1 \Longleftrightarrow \begin{cases} \dot{z} &= a(z,y)w^2 + b(z,y)w + c(z,y) \\ \dot{y} &= w \end{cases}, w \in \mathbb{R}$$

$$\Longleftrightarrow \begin{cases} \dot{z} &= a(z,y)w^2 + b(z,y)w + c(z,y) \\ \dot{y} &= w \\ \dot{w} &= u \end{cases} \quad u \in \mathbb{R} \qquad (\Sigma_q^1)$$

$u \in \mathbb{R}$ is called the *control*, and $\bar{x} = (z, y, w) \in \mathcal{X} \times \mathbb{R}$ is the extended coordinate system.

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Transformations diagram

What is the notion of equivalence for affine systems that make this diagram commute ?

$$
\begin{tikzcd}
\mathcal{F} \arrow[r, leftrightarrow, "\phi"] \arrow[d, leftrightarrow] & \mathcal{F}_q^1 \arrow[d, leftrightarrow] \\
\left\{ \begin{array}{ll} \dot{Z} & = \xi(Z, Y, W) \\ \dot{Y} & = W \\ \dot{W} & = U \end{array} \right. \arrow[r, leftrightarrow, "?"] & \left\{ \begin{array}{ll} \dot{z} & = a(x)w^2 + b(x)w + c(x) \\ \dot{y} & = w \\ \dot{w} & = u \end{array} \right.
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Feedback equivalence

We consider $\Sigma^i : \dot{\bar{x}} = f^i(\bar{x}) + g^i(\bar{x})u^i$ with $u^i \in \mathbb{R}$, $i = 1, 2$.

Definition ((Affine) Feedback Equivalence)

We say that two affine control systems Σ^1 and Σ^2 are feedback equivalent if and only if there exist smooth functions $\alpha(x)$ and $\beta(x)$, $\beta(\cdot) \neq 0$, and a diffeomorphism ϕ of $\mathcal{X}$ such that:

$$f^2 = \frac{\partial \phi}{\partial x}\left(f^1 + \alpha g^1\right),$$

$$g^2 = \frac{\partial \phi}{\partial x}\left(g^1 \beta\right).$$

It is like taking the control $u^1 = \alpha + \beta u^2$. Geometrically, it is the equivalence of affine distributions $\mathcal{A}^2 = \phi_\star \mathcal{A}^1$ where $\mathcal{A}^i = f^i + \mathrm{sp}\left\{g^i\right\}$.

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Main result

Theorem (Affine feedback quadratization)

Let Σ be an affine control system on a 3-dimensional smooth manifold with 1 control. Σ is locally around $\bar{x}_0$ affine feedback equivalent to Σ_q^1 if, and only if,

1. $g \wedge ad_f g \wedge [g, ad_f g]\,(\bar{x}_0) \neq 0$,
2. *The structure functions a and b in the decomposition* $[g,[g,ad_f g]] = a(\bar{x})\,[g, ad_f g] + b(\bar{x}) ad_f g \mod sp\{g\}$ *satisfy*

$$9b + 2a^2 - 3L_g a = 0.$$

These conditions are checkable by algebraic operations and derivations.

Idea behind the proof

If, $g \wedge \mathrm{ad}_f g(\bar{x}_0) \neq 0$ then an affine control system is feedback equivalent to,

$$\begin{cases} \dot{z} & = \xi(x, y, w) \\ \dot{y} & = w \\ \dot{w} & = u \end{cases} , \quad f = \begin{pmatrix} \xi \\ w \\ 0 \end{pmatrix}, \; g = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Observe that if $[g, [g, \mathrm{ad}_f g]] = 0$ (i.e. $\frac{\partial^3 \xi}{\partial w^3} = 0$) then the system is quadratic in this coordinate system.
The idea of the proof is to see how $[g, [g, \mathrm{ad}_f g]] = 0$ is transformed under the feedback transformations (α, β).

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The theorem for sub-manifolds

The theorem is stated for affine control systems. When directly considering the parametrization of a sub-manifold,

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we have the simplification of the conditions,

1. $\xi^{(2)}(\bar{x}_0) \neq 0$,
2. $a(x) = \frac{\xi^{(3)}}{\xi^{(2)}}$, $b = 0$ and the relation reads,

$$2\left(\frac{\xi^{(3)}}{\xi^{(2)}}\right)^2 - 3\left(\frac{\xi^{(3)}}{\xi^{(2)}}\right)' = 0$$

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Classification of quadratic sub-manifolds

Going back to our sub-manifolds. Once we have distinguished a special class of sub-manifolds, we wan to exhibit normal forms for that class. For example, for a linear sub-manifold $\omega(\dot{x}) = 0$ the problem of classification is the problem of classification of distributions.

In our case, we have:

$$\dot{z} = a(x)\dot{y}^2 + b(x)\dot{y} + c(x). \qquad (\mathcal{F}_q)$$

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Parametrization and equivalence

To deal with this problem, we consider the parametrization of $\mathcal{F}_q^1$ given by

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here, w play the role of control and Ξ_q^1 can be seen as a nonlinear control system. We act on Ξ_q^1 by *diffeomorphisms* $\tilde{x} = \phi(x)$ and *reparametrization* (they are nonlinear feedback) $\tilde{w} = \psi(z, y, w)$ (with $\frac{\partial \psi}{\partial w} \neq 0$).
Since we have to preserve the quadratic structure, we allow reparametrizations of the shape $\tilde{w} = \beta(z, y)w$ only. We identify the vector fields $A = a(x)\frac{\partial}{\partial z}$ and $B = b(x)\frac{\partial}{\partial z} + \frac{\partial}{\partial y}$.

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$$\begin{cases} \dot{z} &= a(x)w^2 + \quad b(x)w + c(x) \\ \dot{y} &= \qquad\qquad\qquad w \end{cases} \qquad (\Xi_q^1)$$

here, w play the role of control and Ξ_q^1 can be seen as a nonlinear control system. We act on Ξ_q^1 by *diffeomorphisms* $\tilde{x} = \phi(x)$ and *reparametrization* (they are nonlinear feedback) $\tilde{w} = \psi(z, y, w)$ (with $\frac{\partial \psi}{\partial w} \neq 0$).
Since we have to preserve the quadratic structure, we allow reparametrizations of the shape $\tilde{w} = \beta(z, y)w$ only. We identify the vector fields $A = a(x)\frac{\partial}{\partial z}$ and $B = b(x)\frac{\partial}{\partial z} + \frac{\partial}{\partial y}$.

Parametrization and equivalence

Notice that since $a(x_0) \neq 0$ we have $A \wedge B \neq 0$. We call (A, B) a *frame*.

Structure of the transformations

On (A, B) the reparametrization, $\tilde{w} = \beta w$, acts by

$$\tilde{A} = \beta^2 A, \quad \tilde{B} = \beta B.$$

Observe that if $a = 1$ and $b = 0$ then $[A, B] = 0$.

The **question** is then when does a reparametrization exist such that $\left[\tilde{A}, \tilde{B}\right] = 0$?

Main result

Theorem

There exists a diffeomorphism and a reparametrization such that $\tilde{a} = 1$ and $\tilde{b} = 0$ if, and only if,

$$[A, [A, B]] = 0, \Longleftrightarrow \frac{\partial}{\partial z}\left(a\frac{\partial}{\partial z}\left(\frac{b}{a}\right)\right) = 0.$$

Moreover, c is an invariant of the sub-manifold.

Then we have:

$$\begin{array}{rcccl} c = 0 & \Longleftrightarrow & \tilde{c} = 0 & & \\ \mathrm{L}_A c = 0 & \Longleftrightarrow & \mathrm{L}_{\tilde{A}}\tilde{c} = 0 & \Longleftrightarrow & \tilde{c}(\tilde{z}, \tilde{y}) = \gamma(\tilde{y}) \\ \mathrm{L}_B c = 0 & \Longleftrightarrow & \mathrm{L}_{\tilde{B}}\tilde{c} = 0 & \Longleftrightarrow & \tilde{c}(\tilde{z}, \tilde{y}) = \gamma(\tilde{z}) \end{array}$$

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Conclusion and perspectives

We presented an introduction to the equivalence and classification problem of *quadratic* sub-manifolds in $\mathrm{T}\mathbb{R}^2$.

There is a way that directly gives necessary and sufficient conditions for the equivalence of Σ with $\dot{z} = w^2$. This is done by the study of the Lie algebra of infinitesimal symmetries, and is easily generalisable in higher dimension (however when $m > 2$, checkablity of the conditions is hard).

We have results for the equivalence and classification of quadratic sub-manifolds in $\mathrm{T}\mathbb{R}^3$:

$$\dot{z} = a(x)(\dot{y}_0^2 + \epsilon \dot{y}_1^2) + b_0(x)\dot{y}_0 + b_1(x)\dot{y}_1 + c(x).$$

The case when $\epsilon = -1$ is called hyperbolic and is *easy* to solve (the geometry is nice). The case when $\epsilon = 1$ is called elliptic, and is a bit more trickier to deal with.
The case when $\epsilon = 0$ (i.e parabolic) is still resisting to us.

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